Exam Special Relativity
December 10, 2015
Start: 9:00h End: 11:00h

Each problem on separate sheet, with your name and student ID

INSTRUCTIONS: This is a closed-book and closed-notes exam. The exam duration is 2 hours. You are allowed to use a numerical (non programmable) calculator. There is a total of 9 points that you can collect. Problems are designed, as much as it is possible, so that you can answer a given part of the problem without necessarily answer other parts.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also loose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need. Write clearly, and draw clearly the spacetime diagrams making use of the squared paper. **Solve the problems on separate sheets** (1 sheet has 4 sides) e.g. problem 1 on sheet 1 and problem 2 on sheet 2.

USEFUL FORMULAS AND CONSTANTS

To convert between SI and SR units use the speed of light $c = 3.00 \times 10^8$ m/s

$$E = \frac{m}{\sqrt{1-v^2}} \qquad P_x = \frac{mv_x}{\sqrt{1-v^2}} \qquad P_y = \frac{mv_y}{\sqrt{1-v^2}} \qquad P_z = \frac{mv_z}{\sqrt{1-v^2}}$$

Lorentz transformation from the Home Frame (t, x, y, z) to the Other Frame (t', x', y', z')

$$\Delta t' = \gamma \Delta t - \gamma\beta\Delta x \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\Delta x' = \gamma \Delta x - \gamma\beta\Delta t$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$

$$L_R = \gamma L$$

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

1. (6 points total) A source at the CERN laboratory in Geneva emits two particles (call this event A) at a certain time in the laboratory frame. Choose event A as the origin of the laboratory frame. One of the two emitted particles, call it P1, travels in the +x direction of the laboratory frame with speed $\beta_1 = 3/5$ and reaches detector D1 located 9 km from the source (call it event B). The other particle, call it P2, travels with speed $\beta_2 = 1/3$ in the opposite direction and reaches a second detector D2 located 3 km from the source (call it event C). At the same instant, detector D2 emits a flash of light that shortly after reaches detector D1 (call it event D). Work in SR units.

a) (1 points) When do events B, C and D occur in the laboratory frame? Does particle P2 reach detector D2 before particle P1 reaches detector D1, according to laboratory clocks? Is the spacetime interval between events B and C timelike, lightlike or spacelike? **Show your work**.

Answer: We first convert distances to SR units:

$$\begin{aligned} \frac{9\,\mathrm{k}m}{3 \times 10^5\,\mathrm{k}m/s} &= 3 \times 10^{-5}\,\mathrm{s} = 30\,\mu\mathrm{s} \\ \frac{3\,\mathrm{k}m}{3 \times 10^5\,\mathrm{k}m/s} &= 10^{-5}\,\mathrm{s} = 10\,\mu\mathrm{s} \end{aligned} \tag{1}$$

Event B occurs at coordinate time $t_B = (5/3) \times 30\mu\mathrm{s} = 50\mu\mathrm{s}$ in the laboratory frame, using that distance=velocity $\times$ time. Analogously, event C occurs at coordinate time $t_C = (-3) \times (-10\mu\mathrm{s}) = 30\mu\mathrm{s}$. Given that a light flash travels at the speed of light $c = 1$, the coordinate time interval between event C and event D equals the distance between detectors D1 and D2 in SR units, i.e., 40μs. Hence, $t_D = t_C + 40\mu\mathrm{s} = 70\mu\mathrm{s}$.

$t_C < t_B$, therefore yes, particle P2 reaches detector D2 before particle P1 reaches D1, according to laboratory clocks.

The (squared) spacetime interval between B and C is given by $\Delta s^2 = (20\mu s)^2 - (40\mu s)^2 < 0$. It is spacelike, with $\Delta t^2 < \Delta x^2$. This also means that events B and C are not causally connected.

b) (1 points) Draw a two-observer spacetime diagram of the situation, where the Home Frame is that of the laboratory and the Other Frame is that of particle P1. **Carefully draw and calibrate** the t' and x' axes for the particle P1 frame. **Show your work**.

To calibrate the t' and x' axis we use the factor $\gamma = (1 - \beta_1^2)^{-1/2} = 5/4 = 1.25$, the relation $\Delta t = \gamma \Delta t'$ for an event on the t' axis with coordinates $\Delta t'$ and $\Delta x' = 0$, and the analogous relation $\Delta x = \gamma \Delta x'$ for an event on the x' axis with coordinates $\Delta t' = 0$ and $\Delta x'$: thus $t' = 1\,\mu$s has coordinate time $t = 1.25\,\mu$s in the Home Frame, and $x' = 1\,\mu$s has spatial coordinate $x = 1.25\,\mu$s in the Home Frame. See Figure 1.

c) (1.5 points) **Draw and label** the worldlines of detectors D1 and D2, particle P1, particle P2 and the flash of light (use a long-dash line for the flash of light). **Locate and label** events A, B, C and D as points on the diagram.

Answer:

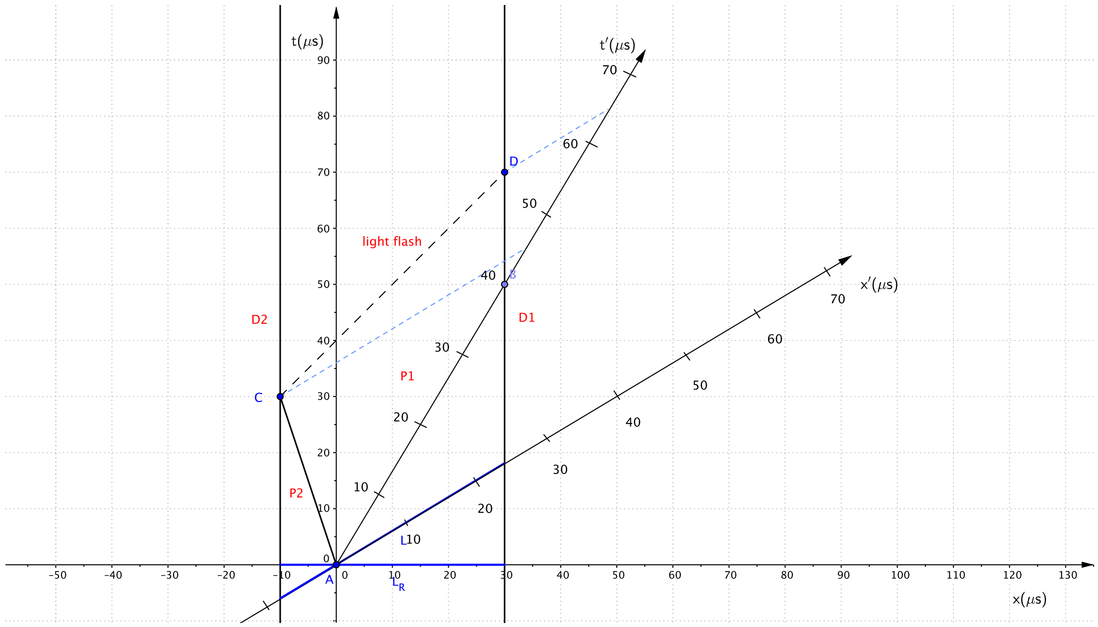

Figure 1: Two-observer diagram, where the Home Frame is the laboratory frame and the Other Frame is the particle P1 frame. We have chosen the origins of both frames at event A. **NB You should add ticks on the t' and x' axes, spaced by 5μs or less in order to have sufficient accuracy to read the value of L.**

d) (1.5 points) When do events B, C and D occur in the particle P1 frame? Use the algebraic method with the appropriate Lorentz transformation equations and **compare your result** with the approximate one obtained by using the graphical method. **Show your work**. A sensor attached to particle P1 registers that in its own frame particle P1 reaches detector D1 before particle P2 reaches detector D2. Is this true? Is this causing a violation of causality? **Show your work**.

Answer: The Lorentz transformation equations from the Home Frame to the Other Frame yield

$$\begin{aligned} t'_C &= \gamma t_C - \gamma\beta_1 x_C = 45\mu s \\ t'_B &= \gamma t_B - \gamma\beta_1 x_B = 40\mu s \\ t'_D &= \gamma t_D - \gamma\beta_1 x_D = 65\mu s\,, \end{aligned} \tag{2}$$

where $x_B = 30\,\mu$s is the spatial coordinate of D1 in the laboratory frame with origin at the source, $x_C = -10\,\mu$s is the spatial coordinate of D2, and $x_D = x_B$. These values are in good agreement with the approximate ones obtained graphically by drawing parallels to the x' axis from events B, C and D and reading the coordinate of the intersection with the t' axis. We read $t'_B \approx 40\mu$s, $t'_C \approx 45\mu$s and $t'_D \approx 65\mu$s.

We conclude that $t'_C > t'_B$, therefore what the sensor registers is true: P1 reaches D1 before P2 reaches D2 in P1 frame. No violation of causality occurs, because events B and C are not causally connected, as shown in a). In other words, an inertial reference frame exists where the two events are simultaneous, and inertial reference frames exist where their temporal ordering is reversed.

e) (1 points) What is the distance between detectors D1 and D2 as measured in the particle P1 frame? **Indicate the distances** in the Home Frame and in the Other Frame **on the two-observer diagram**. Use the algebraic method with the appropriate equations and **compare your result** with the graphical method.

Lorentz contraction in the moving frame tells that the distance between D1 and D2 of $L_R = 40\,\mu$s as measured in their rest frame is contracted to $L = L_R/\gamma = 40\mu\text{s}(4/5) = 32\mu\text{s}$ in the P1 frame. This is in good agreement with what we read on the two-observer diagram, where L_R and L (indicated on the diagram) are the spatial separation between two events, one on D2 worldline and one on D1 worldline that are *simultaneous* in the laboratory frame and the P1 particle frame, respectively.

2. (3 points total) An object with a mass m sits at rest. A light flash moving in the $+x$ direction with a total energy of $3m/2$ hits this object and is completely absorbed. What are the mass, the speed and the velocity vector's components of the object afterward?

a) (1 points) Answer this question approximately, using an energy-momentum diagram.

Answer: The process happens along the x axis, therefore we can conveniently describe it with an energy-momentum diagram in E and P_x; see Figure 2. $\mathbf{P}_1$ is the four-momentum of the light flash, $\mathbf{P}_2$ the four-momentum of the object with mass m and $\mathbf{P}_T$ the final four-momentum of the object with mass M. On the diagram, we read the P_x and E components of $\mathbf{P}_T$, $P_x = 1.5m$ and $E = 2.5m$ with rather good precision. The non-zero velocity component v_x is determined by the slope of P_T, i.e., $v_x = P_x/E = 1.5/2.5 = 0.6$ and $v_y = v_z = 0$. The mass of the final object M identifies the hyperbola $E^2 - P_x^2 = M^2$. One point on this hyperbola, i.e., $P_x = 1.5m$ and $E = 2.5m$ determines $M^2 = (2.5m)^2 - (1.5m)^2 = 4m^2$, thus $M = 2m$. Equivalently, M is the intersection of the hyperbola with the E axis.

b) (2 points) Solve this problem quantitatively, using four-dimensional vectors. **Show your work**.

Answer: $M = 2m$, $v = 3/5$, $\vec{v} = (3/5, 0, 0)$.

The conservation of four-momentum reads

$$\begin{bmatrix} M/\sqrt{1-v^2} \\ Mv_x/\sqrt{1-v^2} \\ Mv_y/\sqrt{1-v^2} \\ Mv_z/\sqrt{1-v^2} \end{bmatrix} = \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{2}m \\ \frac{3}{2}m \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

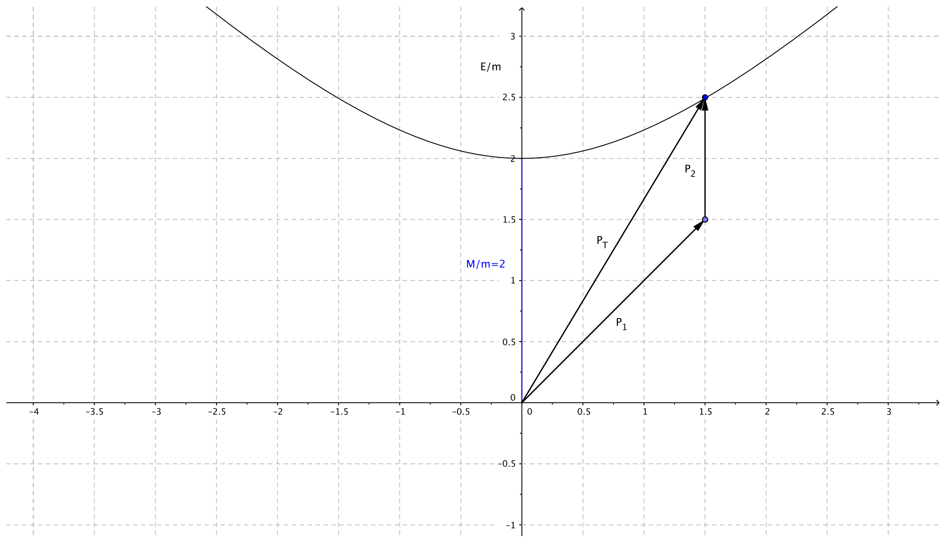

Figure 2: Energy-momentum diagram for the process described in this problem. Energy and x-component of the momentum P_x are measured in units of m.

where M is the mass of the final object and $\vec{v} = (v_x, v_y, v_z)$ its velocity. The equation for the y and z components imply that $v_y = v_z = 0$, thus $v^2 = v_x^2$, and the first two equations yield

$$\begin{aligned} \frac{M}{\sqrt{1-v_x^2}} &= m + \frac{3}{2}m \\ \frac{M v_x}{\sqrt{1-v_x^2}} &= \frac{3}{2}m \end{aligned} \tag{4}$$

Divide the second equation by the first to obtain $v_x = 3/5$. Square both sides of both equations and subtract the second one from the first one to obtain $M = 2m$.